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Non-perturbative solution of a quantum mechanical oscillator interacting with a specific environment†

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Abstract. A quantum mechanical model of an oscillator interacting linearly with an environment is treated by the method of perturbation series expansion. For a special class of environments and interactions, this series is summed up to all orders. An integral equation for the time dependence of the coordinate operator of the oscillator is obtained, which is solved analytically by the method of Laplace transformations. General conditions are stated for a dissipative behaviour of the special class of environments considered. An example, which is widely applicable, is discussed.

1. Introduction

The quantum mechanical problem of an oscillator interacting with an environment has obtained much attention in literature. A recent review of the status of the field has been given by Dekker (1981). One method to attack the problem is to derive quantum mechanical Langevin equations of motion for the oscillator coordinate and momentum in the Heisenberg picture by stating explicitly the interaction with the environment. Models which have been solved exactly consist of an oscillator-type (boson-type) environment and an interaction that is linear in the coordinates and momenta of the system oscillator and the environment (see e.g. Schwabl and Thirring 1964, Ford *et al* 1965, and more recently Arai 1981, Maassen 1982, Alicki 1982).

In several studies the dissipative nature of the environment is treated in perturbation theory up to second order only (e.g. Senitzky 1960, Hasse 1979). In this paper we show that under certain specific conditions for the environment and the interaction the perturbation series can be summed up to all orders, and then Langevin-type equations of motion for the oscillator coordinate are obtained. It will be shown that the condition for the summability of the perturbation series defines a class of environments and interactions which comprehend the cases of exactly solvable models mentioned above.

The paper is organised as follows. In §§ 2 and 3 we introduce the model and calculate the time development of the oscillator coordinate by the perturbation expansion method, respectively. In § 4 we state the condition for the summability of the perturbation series. The integral equation for the oscillator coordinate and its solution are discussed in §§ 5–9, especially for the weak damping case (§ 7) and in the framework of the short time interaction approach (§ 8). Our conclusions are given in § 10.

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2. The model

We consider an oscillator interacting with an environment. The Hamiltonian assumed is of the form (Schrödinger picture)

$$H = H_0 + \lambda W, \tag{1}$$

where

$$H_0 = H_{\text{osc}}(P, Q) + H_{\text{env}}(p, x) \tag{2}$$

$$H_{\text{osc}} = P^2/2M + \tfrac{1}{2}M\Omega^2 Q^2 \tag{3}$$

$$W = Q \cdot V(p, x). \tag{4}$$

Here, H_0 is the unperturbed Hamiltonian consisting of the system oscillator and the environment Hamiltonians. The coordinate and momentum of the oscillator are denoted by Q and P, respectively, with $[Q, P] = i\hbar$. The set of coordinates and momenta of the environment is denoted by (x, p). The interaction operator W is multiplied by a dimensionless perturbation parameter λ and is assumed linear in Q. In the Heisenberg picture the time development of the nth power of the operator Q is given by

$$Q^n(t) = \exp(iHt/\hbar) Q^n \exp(-iHt/\hbar). \tag{5}$$

Since one is usually interested in the time dependence of the mean value $\langle Q \rangle$ and the variance $\langle Q^2 \rangle - \langle Q \rangle^2$ under the influence of an environment, we study in the following the evaluation of equation (5) under certain stringent conditions for the environment, where an analytical expression for $Q(t)$ and $Q^2(t)$ can be obtained.

3. Perturbation expansion of the coordinate $Q(t)$

The time-development operator in the intermediate representation

$$U(t) = \exp(iH_0 t/\hbar) \exp(-iHt/\hbar) \tag{6}$$

fulfils the integral equation (Messiah 1961)

$$U(t) = 1 - \frac{i}{\hbar}\lambda \int_0^t \tilde{W}(t') U(t')\, dt' \tag{7}$$

where

$$\tilde{W}(t) = \exp(iH_0 t/\hbar) W \exp(-iH_0 t/\hbar) = \tilde{Q}(t)\tilde{V}(t) \tag{8}$$

with

$$\tilde{Q}(t) = \exp(iH_0 t/\hbar) Q \exp(-iH_0 t/\hbar) = Q \cos \Omega t + (P/M\Omega) \sin \Omega t \tag{9}$$

$$\tilde{V}(t) = \exp(iH_0 t/\hbar) V \exp(-iH_0 t/\hbar). \tag{10}$$

Solving equation (7) iteratively we obtain the perturbation expansion

$$U(t) = 1 + \sum_{n=1}^{\infty} \left(\frac{-i\lambda}{\hbar}\right)^n \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} w(\tau_1 \ldots \tau_n)\, d\tau^n \tag{11}$$

where

$$d\tau^n = d\tau_1 \ldots d\tau_n \qquad w(\tau_1 \ldots \tau_n) = \tilde{W}(\tau_1) \ldots \tilde{W}(\tau_n). \tag{12}$$

This result is inserted into

$$Q(t) = U^+(t)\tilde{Q}(t)U(t), \tag{13}$$

where $U^+(t)$ is the Hermitian conjugate operator of $U(t)$. Then we get

$$Q(t) = \sum_{n=0}^{\infty} Q^{[n]}(t), \tag{14}$$

where $Q^{[0]} = \tilde{Q}(t)$ and for the nth order of perturbation

$$\begin{aligned} Q^{[n]}(t) = \left(\frac{i\lambda}{\hbar}\right)^n \Bigg(& \int d\tau^n \, w(\tau_n \ldots \tau_1)\tilde{Q}(t) \\ & + \sum_{\substack{m=1 \\ (n\geq 2)}}^{n-1} (-1)^m \int d\tau^{n-m} \, d\theta^m \, w(\tau_{n-m} \ldots \tau_1)\tilde{Q}(t) w(\theta_1 \ldots \theta_m) \\ & + (-1)^n \int d\theta^n \, \tilde{Q}(t) w(\theta_1 \ldots \theta_n) \Bigg) \end{aligned} \tag{15}$$

with the following conditions on the integration variables:

$$t > \tau_1 > \ldots > \tau_{n-m} > 0, \qquad t > \theta_1 > \ldots > \theta_m > 0. \tag{16}$$

For $n \geq 2$ we can reorder the operators and integration boundaries in (15) and get after some simple steps:

$n = 2$:

$$Q^{[2]} = (i\lambda/\hbar)^2 \int d\tau^2 [[\tilde{Q}(t), \tilde{W}(\tau_1)], \tilde{W}(\tau_2)]$$

$n \geq 3$:

$$\begin{aligned} Q^{[n]} = \left(\frac{i\lambda}{\hbar}\right)^n \Bigg(& \int d\tau^n \, w(\tau_n \ldots \tau_3)[[\tilde{Q}(t), \tilde{W}(\tau_1)], \tilde{W}(\tau_2)] \\ & + \sum_{\substack{m=1 \\ (n\geq 4)}}^{n-3} (-1)^m \int d\tau^{n-m} \, d\theta^m \, w(\tau_{n-m} \ldots \tau_3) \\ & \times [[\tilde{Q}(t), \tilde{W}(\tau_1)], \tilde{W}(\tau_2)] w(\theta_1 \ldots \theta_m) \\ & + (-1)^n \int d\tau^2 \, d\theta^{n-2} [[\tilde{Q}(t), \tilde{W}(\tau_1)], \tilde{W}(\tau_2)] w(\theta_1 \ldots \theta_{n-2}) \Bigg) \end{aligned} \tag{17}$$

where now

$$t > \tau_1 > \tau_2 \ldots > \tau_{n-m} > 0 \qquad \tau_2 > \theta_1 > \ldots > \theta_m > 0.$$

The double commutator in (17) can be worked out by using (8) and (9):

$$[[\tilde{Q}(t), \tilde{W}(\tau_1)], \tilde{W}(\tau_2)] = -(i\hbar/M\Omega) \sin \Omega(t - \tau_1)[\tilde{V}(\tau_1), \tilde{V}(\tau_2)]\tilde{Q}(\tau_2). \tag{18}$$

4. Condition of summability of the perturbation series and the choice of the environment and the interaction

In order to proceed further with the perturbation expansion of the oscillator coordinate given by equations (14) and (17), we introduce in (18) the following condition for

the commutator:

$$[\tilde{V}(\tau_1), \tilde{V}(\tau_2)] = f(\tau_1 - \tau_2), \tag{19}$$

where f is an odd c-number function of its argument and independent of the momenta and coordinates of the environment. Insertion of equation (19) into (18) makes (17) a recurrence relation. Before we do this in § 5, in the following we show that the requirement (19) actually defines a special class of Hamiltonians $H_{\text{env}}(p, x)$ and interactions $V(p, x)$ of the environment.

Taking $\tau_1 = t$, $\tau_2 = 0$ in (19) and making a Taylor series expansion of the time-development operator, we can rewrite equation (19) as follows:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n \sum_{m=0}^{n} (-)^m \binom{n}{m} [H_{\text{env}}^{n-m} V H_{\text{env}}^{m}, V] = f(t). \tag{20}$$

The last relation can only be fulfilled if the sum over m yields a constant number, say c_n, independent of time ($n = 1, 3, 5 \ldots$):

$$\sum_{m=0}^{n} (-)^m \binom{n}{m} [H_{\text{env}}^{n-m} V H_{\text{env}}^{m}, V] = c_n. \tag{21}$$

Equations (21) determine the class of possible forms of H_{env} and V. Let us assume that V depends only on the coordinates of the intrinsic degrees of freedom and that H_{env} consists of the kinetic energy T, quadratic in the momenta, and a potential energy U_{env} ($x = x_1, \ldots, x_N$):

$$H_{\text{env}} = \sum_{i=1}^{N} \frac{1}{2m_i} p_i^2 + U_{\text{env}}(x). \tag{22}$$

Then the only possible forms of V and U_{env}, which fulfil equations (21), are the following:

$$V = \sum_{i=1}^{N} v_i x_i + V_0 \tag{23}$$

$$U_{\text{env}} = \sum_{i,j=1}^{N} b_{ij} x_i x_j + \sum_{i=1}^{N} d_i x_i + U_0 \tag{24}$$

where v_i, V_0, b_{ij}, d_i and U_0 are constants.

Equations (23) and (24) together with (22) describe an environment consisting of N harmonic oscillators which couple linearly to the first considered oscillator described by H_{osc}. A more general class of couplings and environments can be generated by carrying out canonical transformations of the intrinsic coordinates, which leave the commutator (19) unchanged.

For illustration, let us assume that H_{env}, given by (22) and (24), is transformed to the following form:

$$H_{\text{env}} = \sum_{i=1}^{N} \left(\frac{1}{2m_i} p_i^2 + \frac{m_i \omega_i^2}{2} x_i^2\right) \tag{25}$$

and

$$V = \sum_{i=1}^{N} a_i x_i. \tag{26}$$

Then the commutator (19) can be easily calculated. We get the simple expression:

$$f(t) = -\mathrm{i}\hbar \sum_{i=1}^{N} \frac{a_i^2}{m_i \omega_i} \sin \omega_i t. \tag{27}$$

Here we want to stress that the formulae derived in the following sections are justified only for couplings and environments obeying equation (19).

5. Integral equations for $Q(t)$

When we insert (19) into (18) and then (18) into (17) and compare the resulting expression with (15), we obtain a recurrence relation connecting $Q^{[n]}$ with $Q^{[n-2]}$. We sum up the recurrence relations for n even and n odd, separately,

$$Q^{\mathrm{even}}(t) = \sum_{n=0}^{\infty} Q^{[2n]}(t), \qquad Q^{\mathrm{odd}}(t) = \sum_{n=0}^{\infty} Q^{[2n+1]}(t), \tag{28}$$

which result in the following two integral equations:

$$Q^{\mathrm{even}}(t) = \tilde{Q}(t) + \frac{\mathrm{i}\lambda^2}{M\Omega\hbar} \int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \sin \Omega(t-\tau_1) f(\tau_1 - \tau_2) Q^{\mathrm{even}}(\tau_2) \tag{29}$$

$$Q^{\mathrm{odd}}(t) = -\frac{\lambda}{M\Omega} \int_0^t \tilde{V}(\tau) \sin \Omega(t-\tau)\, \mathrm{d}\tau + \frac{\mathrm{i}\lambda^2}{M\Omega\hbar} \int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \sin \Omega(t-\tau_1) f(\tau_1 - \tau_2) Q^{\mathrm{odd}}(\tau_2). \tag{30}$$

From these equations it is interesting to observe that Q^{even} is acting in the space of the oscillator and Q^{odd} in the space of the environment only. Adding up equations (29) and (30) we obtain the integral equation for $Q(t)$.

We note that inserting $Q(t)$ into the equation of motion

$$\mathrm{d}^2 Q(t)/\mathrm{d}t^2 + \Omega^2 Q(t) = -(\lambda/M) V(t), \tag{31}$$

where

$$V(t) = \exp(\mathrm{i}Ht/\hbar)\, V \exp(-\mathrm{i}Ht/\hbar), \tag{32}$$

gives us another interesting integral equation connecting $V(t)$ with $Q(t)$:

$$V(t) = \tilde{V}(t) - \frac{\mathrm{i}\lambda}{\hbar} \int_0^t f(t-\tau) Q(\tau)\, \mathrm{d}\tau. \tag{33}$$

Equations (31) and (33) yield the following integro-differential equation for $Q(t)$:

$$\frac{\mathrm{d}^2 Q(t)}{\mathrm{d}t^2} + \Omega^2 Q(t) = -\frac{\lambda}{M} \tilde{V}(t) + \frac{\mathrm{i}\lambda^2}{M\hbar} \int_0^t f(t-\tau) Q(\tau)\, \mathrm{d}\tau. \tag{34}$$

6. General solution for $Q(t)$

Equations (29) and (30) or, equivalently, equation (34) can be solved with the method of Laplace transformations. The Laplace transform is defined for a time-dependent

function $\varphi(t)$ as (Bronstein and Semendjajew 1967)

$$L\varphi(s)=\int_0^\infty \exp(-st)\varphi(t)\,\mathrm{d}t \tag{35}$$

with $\mathrm{Re}\,(s)>0$.

For the Laplace transform of $Q(t)$ we find from equation (34):

$$LQ(s)=[s^2+\Omega^2-(\mathrm{i}\lambda^2/M\hbar)Lf(s)]^{-1}[P/M+Qs-(\lambda/M)L\tilde{V}(s)]. \tag{36}$$

Then transforming back to $Q(t)$ we obtain the general solution:

$$Q(t)=A(t)\frac{P}{M}+B(t)Q-\frac{\lambda}{M}\int_0^t A(t-\tau)\tilde{V}(\tau)\,\mathrm{d}\tau, \tag{37}$$

where $P=P(t=0)$ and $Q=Q(t=0)$ are the operators of the momentum and coordinate of the system oscillator at time $t=0$. At $t=0$ these operators coincide in the Heisenberg and Schrödinger pictures according to the definition (5). The Laplace transforms of $A(t)$ and $B(t)$ are given by

$$LA(s)=[s^2+\Omega^2-(\mathrm{i}\lambda^2/M\hbar)Lf(s)]^{-1} \tag{38}$$

$$LB(s)=sLA(s). \tag{39}$$

In order that the environment acts dissipatively on the system oscillator we require that $Q(t)$ approaches the inhomogeneous solution of (34) for large times:

$$\lim_{t\to\infty} Q(t)=-\frac{\lambda}{M}\int_0^t A(t-\tau)\tilde{V}(\tau)\,\mathrm{d}\tau. \tag{40}$$

This condition is in general fulfilled if $LA(s)$ is a meromorphic function in the complex s plane with poles at $s=s_1, s_2\ldots$ and $\mathrm{Re}\,(s_i)<0$, and is determined by the specific form of $Lf(s)$. For the oscillator-type environment, given by (25) and (26), we obtain for $Lf(s)$, using (27):

$$Lf(s)=-\mathrm{i}\hbar\sum_{i=1}^{N}\frac{a_i^2}{m_i}\frac{1}{s^2+\omega_i^2}. \tag{41}$$

Then the Laplace transformed function

$$LA(s)=\left(s^2+\Omega^2-\frac{\lambda^2}{M}\sum_{i=1}^{N}\frac{a_i^2}{m_i}\frac{1}{s^2+\omega_i^2}\right)^{-1} \tag{42}$$

has poles in the complex s plane for $s^2=x_j$ $(j=1,\ldots,N+1)$, where the x_j are positive and negative real numbers. Since all poles, except possibly two, have real parts $\mathrm{Re}\,(s_i)=0$, the considered environment does not behave dissipatively.

A dissipative oscillator-type environment is obtained if we assume a continuous frequency spectrum. Setting the density of eigenmodes $\rho(\omega)$ and the frequency-dependent coupling constant $a(\omega)$ and mass $m(\omega)$ we can write:

$$f(t)=\int_{-\infty}^{+\infty}\tilde{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{43}$$

$$\tilde{f}(\omega)=-\tfrac{1}{2}\hbar a^2(\omega)\rho(\omega)/(m(\omega)\cdot\omega) \tag{44}$$

$$Lf(s)=\int_{-\infty}^{+\infty}\tilde{f}(\omega)\frac{1}{s-\mathrm{i}\omega}\,\mathrm{d}\omega. \tag{45}$$

Here, we have assumed $\rho(\omega)$, $a(\omega)$ and $m(\omega)$ as even functions in ω. Therefore the Fourier transformed function $\tilde{f}(\omega)$ is an odd function, $\tilde{f}(\omega) = -\tilde{f}(-\omega)$, in ω. Further let us assume that $\tilde{f}(\omega)$ is a meromorphic function in the complex ω plane with no poles at the real ω axis and a zero at infinity. Then the function $f(t)$ is defined for $t > 0$ by the poles of $\tilde{f}(\omega)$ in the upper half of the complex ω plane:

$$f(t) = 2\pi i \sum_k \operatorname{Res}(\tilde{f}(\omega)\, e^{i\omega t})|_{\omega = \alpha_k + i\beta_k}. \tag{46}$$

Here $\alpha_k + i\beta_k$ with $\beta_k > 0$ denote the positions of the poles in the upper half of the complex ω plane. Depending on the maximum value of the times $1/\beta_k$, the function $f(t)$ acts only up to times of the order $t \sim \max(1/\beta_k)$. For much larger times the function $f(t)$ is negligibly small—a property to be used in connection with the short time interaction approximation (§ 8). The corresponding Laplace transform (45) is then obtained as

$$Lf(s) = -2\pi \sum_k \operatorname{Res}\left(\frac{\tilde{f}(\omega)}{\omega + is}\right)\Bigg|_{\omega = \alpha_k + i\beta_k}. \tag{47}$$

Therefore, $Lf(s)$ is a meromorphic function in the complex s plane with poles at $s = -\beta_k + i\alpha_k$.

As an example we take the following model ($\beta > 0$, $f_0 > 0$):

$$\tilde{f}(\omega) = -(f_0 \omega/\pi)/(\omega^2 + \beta^2). \tag{48}$$

This yields for $f(t)$ and $Lf(s)$:

$$t > 0 \qquad f(t) = -if_0\, e^{-\beta t} \tag{49}$$

$$Lf(s) = -if_0/(s+\beta). \tag{50}$$

Then the Laplace transform (38) of the model becomes

$$LA(s) = [s^2 + \Omega^2 - (\lambda^2 f_0/M\hbar)(s+\beta)^{-1}]^{-1}. \tag{51}$$

This function has three poles of first order in the complex s plane. The coordinates of these three poles have $\operatorname{Re}(s_i) < 0$ if the condition $\Omega^2\beta > \lambda^2 f_0/M\hbar$ is fulfilled. Under this condition the environment behaves dissipatively.

7. Weak damping

For the following we assume that the damping of the system oscillator due to the environment is weak. In that case the function $LA(s)$ has two poles of first order in the vicinity of $s = \pm i\Omega$, besides the possible other poles with $\operatorname{Re}(s_i) \ll 0$. The poles in the vicinity of $s = \pm i\Omega$ are responsible for the long time behaviour of $A(t)$ and $B(t)$, since they yield the smallest decay times. Using the parameter λ as perturbation parameter, the positions of these poles in the complex s plane are obtained up to terms in λ^2:

$$s_1 = r(\Omega), \qquad s_2 = r(-\Omega), \tag{52}$$

where

$$r(\Omega) = i\Omega + (\lambda^2/2M\Omega\hbar) Lf(i\Omega). \tag{53}$$

With the definition of the Laplace and Fourier transforms of $f(t)$ we get:

$$Lf(\pm i\Omega) = (1/2i)(\gamma(\Omega) \mp i g(\Omega)) \tag{54}$$

where $\gamma(\Omega)$ and $g(\Omega)$ are real functions defined by

$$\gamma(\Omega) = 2i \int_0^\infty \cos \Omega t f(t)\, dt \tag{55}$$

$$g(\Omega) = 2i \int_0^\infty \sin \Omega t f(t)\, dt = -2\pi \tilde{f}(\Omega). \tag{56}$$

Inserting (54) and (56) into (52) we obtain finally:

$$r(\pm\Omega) = \pm i\tilde{\Omega} - \tilde{\Gamma} \tag{57}$$

where

$$\tilde{\Omega} = \Omega - (\lambda^2/4M\Omega\hbar)\gamma(\Omega) \tag{58}$$

$$\tilde{\Gamma} = -(\lambda^2\pi/2M\Omega\hbar)\tilde{f}(\Omega). \tag{59}$$

The frequency shift $\Delta\Omega = \tilde{\Omega} - \Omega$ can be positive or negative, whereas the decay width $\tilde{\Gamma}$ is always positive because $\tilde{f}(\Omega)$ is negative for $\Omega > 0$ (see equation (44)). In the model, where $f(t)$ is given by (49), the frequency shift and the decay width are calculated as:

$$\tilde{\Omega} - \Omega = -\frac{\lambda^2 f_0}{2M\hbar} \frac{\beta/\Omega}{\Omega^2 + \beta^2}, \qquad \tilde{\Gamma} = \frac{\lambda^2 f_0}{2M\hbar} \frac{1}{\Omega^2 + \beta^2}. \tag{60}$$

Assuming that the two poles of $LA(s)$ at s_1 and s_2 (equation (52)) determine the long time behaviour of $A(t)$ and $B(t)$ we obtain for these functions ($\mathrm{Re}(s_i) \ll -\tilde{\Gamma}$ for $i > 2$):

$$A(t) = (1/\tilde{\Omega}) \sin \tilde{\Omega} t\, e^{-\tilde{\Gamma} t} \tag{61}$$

$$B(t) = (\cos \tilde{\Omega} t - (\tilde{\Gamma}/\tilde{\Omega}) \sin \tilde{\Omega} t)\, e^{-\tilde{\Gamma} t}. \tag{62}$$

From these functions it follows that in the approximation of weak damping (i.e. equation (53) is correct only up to terms in λ^2), the long time behaviour of $Q(t)$ is described by the differential equation for a damped oscillator with an external driving force:

$$d^2Q(t)/dt^2 + 2\tilde{\Gamma}\, dQ(t)/dt + (\tilde{\Omega}^2 + \tilde{\Gamma}^2)Q(t) = -(\lambda/M)\tilde{V}(t). \tag{63}$$

It is clear that the position of the poles near $\pm i\Omega$ can be calculated from (38) without any restrictions on λ. In that case the two poles do not lie, in general, symmetrically with respect to the real s axis as in the approximation (57), and then the expressions for $A(t)$ and $B(t)$ are slightly more complex as compared with that given by equations (61) and (62).

8. The short time interaction approximation

The function $f(t)$ acts as the response function of the environment. As stated above, it acts only up to times of the order $t \sim \max(1/\beta_k)$ (see equation (46)). This fact can be used for an approximation of the integrals in (29) and (30) for times $t \gg \max(1/\beta_k)$.

First we transform the integrals in (29) and (30) to new variables of integration ($\tau = \tau_2$, $\theta = \tau_1 - \tau_2$):

$$\int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \sin \Omega(t-\tau_1) f(\tau_1-\tau_2) Q(\tau_2)$$
$$= \int_0^t \mathrm{d}\tau\, Q(\tau) \int_0^{t-\tau} \mathrm{d}\theta \sin \Omega(t-\tau-\theta) f(\theta). \tag{64}$$

Next we introduce the following approximation for $t-\tau \gg \max(1/\beta_k)$, which in the literature is known as the short time interaction approach (see Senitzky 1960):

$$2\mathrm{i} \int_0^{t-\tau} \mathrm{d}\theta \sin \Omega(t-\tau-\theta) f(\theta)$$
$$\to 2\mathrm{i} \int_0^{\infty} \mathrm{d}\theta \sin \Omega(t-\tau-\theta) f(\theta)$$
$$= \gamma(\Omega) \sin \Omega(t-\tau) - g(\Omega) \cos(t-\tau). \tag{65}$$

Here $\gamma(\Omega)$ and $g(\Omega)$ have already been defined in equations (55) and (56), respectively. Inserting the approximative expression (65) into (64) and then back into (29) and (30), we obtain the following simplified integral equation for $Q(t)$ in the short time interaction approximation:

$$Q(t) = \tilde{Q}(t) - \frac{\lambda}{M\Omega} \int_0^t \tilde{V}(\tau) \sin \Omega(t-\tau)\, \mathrm{d}\tau$$
$$+ \frac{\lambda^2}{2M\Omega\hbar} \int_0^t (\gamma(\Omega) \sin \Omega(t-\tau) - g(\Omega) \cos \Omega(t-\tau)) Q(\tau)\, \mathrm{d}\tau. \tag{66}$$

With the method of Laplace transformation the time-dependent solution is again easily calculated:

$$Q(t) = A'(t) \frac{P}{M} + B'(t) Q - \frac{\lambda}{M} \int_0^t A'(t-\tau) \tilde{V}(\tau)\, \mathrm{d}\tau \tag{67}$$

where P and Q are defined as in equation (37), and

$$LA'(s) = [s^2 + \Omega^2 + (\lambda^2/2M\Omega\hbar)(g(\Omega)s - \gamma(\Omega)\Omega)]^{-1} \tag{68}$$

$$LB'(s) = sLA'(s). \tag{69}$$

The $LA'(s)$ has two poles of first order, given up to terms in λ^2 exactly by the expression (57):

$$s_{1,2} = \pm \mathrm{i}\tilde{\Omega} - \tilde{\Gamma}, \tag{70}$$

with $\tilde{\Omega}$ and $\tilde{\Gamma}$ defined by (58) and (59). Hence the functions $A'(t)$ and $B'(t)$ have the same time behaviour as $A(t)$ and $B(t)$, studied in § 7 for the case of weak damping and large times. Therefore, the short interaction time approach and the weak damping approximation yield the same results in the framework of our model (up to terms of λ^2 in s_1 and s_2).

9. The mean value of $Q(t)$ and $Q^2(t)$ at large times

In fact, we are interested in the mean values of the operators $Q(t)$ and $Q^2(t)$ over the environment in connection with e.g. charge and mass transfer during nuclear heavy ion reactions (Gupta *et al* 1983). The environment can be considered to be described by either (i) a pure quantum state, say $|n\rangle$, or (ii) a density matrix. In particular, if the environment is represented by a canonical ensemble at temperature T, the density matrix is given by

$$\hat{\rho} = \sum_n |n\rangle p_n \langle n| \tag{71}$$

with

$$p_n = (1/Z)\exp(-E_n/kT). \tag{72}$$

Here $|n\rangle$ and E_n are the eigenstates and eigenvalues, respectively, of the environment Hamiltonian H_{env}. At large times the mean value of $Q(t)$ with respect to the environment, where $Q(t)$ is given by (40), vanishes for the environment and interaction given by (25) and (26), respectively. For the calculation of the mean value of $Q^2(t)$ at large times we take the square of equation (40):

$$Q^2(t) \underset{t\to\infty}{=} \tfrac{1}{2}\left(\frac{\lambda}{M}\right)^2 \int_0^t \int_0^t A(t-\tau_1)A(t-\tau_2)(\tilde{V}(\tau_1)\tilde{V}(\tau_2)+\tilde{V}(\tau_2)\tilde{V}(\tau_1))\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{73}$$

The anticommutator appearing in (73) can be evaluated by taking the expectation value with respect to the environment. The environment is assumed to be described with oscillators distributed with a density $\rho(\omega)$ over the frequencies ω. For an easy calculation of the expectation value of the anticommutator in (73), we first take an environment consisting of oscillators with a discrete frequency spectrum and finally make the transition to the continuous spectrum. Therefore the environment is first described by the Hamiltonian (25) and initially represented by a canonical ensemble at temperature T (see (71) and (72)) and the interaction V between environment and oscillator is given by (26). We then obtain:

$$\tfrac{1}{2}\langle \tilde{V}(\tau_1)\tilde{V}(\tau_2)+\tilde{V}(\tau_2)\tilde{V}(\tau_1)\rangle_{\text{env}} = \frac{\hbar}{2}\sum_{i=1}^{N}\frac{a_i^2}{m_i\omega_i}\coth\left(\frac{\hbar\omega_i}{2kT}\right)\cos\omega_i(\tau_1-\tau_2). \tag{74}$$

Inserting this result back into (73) we carry out the time integrations by using the Laplace transformation of $A(t)$. Then we replace the discrete frequencies of the oscillator environment by the continuous spectrum and make use of the Fourier transformed function of $f(t)$ (see equation (44)). This results finally in

$$\langle Q^2\rangle_{\text{env}} \underset{t\to\infty}{=} -\frac{1}{2}\left(\frac{\lambda}{M}\right)^2 \int_{-\infty}^{+\infty} \tilde{f}(\omega)\coth\left(\frac{\hbar\omega}{2kT}\right) LA(\mathrm{i}\omega)LA(-\mathrm{i}\omega)\,\mathrm{d}\omega, \tag{75}$$

where $LA(s)$ is defined in (38). This expression can be evaluated further only if $\tilde{f}(\omega)$ and $LA(s)$ are explicitly given. In the weak damping limit or, equivalently, in the short interaction time approach, we can use $A(t)$ in the form given by (61). This yields the expression

$$\langle Q^2\rangle_{\text{env}} \underset{t\to\infty}{=} -\frac{1}{2}\left(\frac{\lambda}{M}\right)^2 \int_{-\infty}^{+\infty} \tilde{f}(\omega)\coth\left(\frac{\hbar\omega}{2kT}\right)((\omega^2-\tilde{\Omega}^2-\tilde{\Gamma}^2)^2+4\omega^2\tilde{\Gamma}^2)^{-1}\,\mathrm{d}\omega. \tag{76}$$

In the limit of very weak coupling ($\lambda \to 0$) one obtains the well known result

$$\langle Q^2 \rangle_{\text{env}} \underset{\substack{t\to\infty \\ \lambda\to 0}}{=} \frac{\hbar}{2M\Omega} \coth\left(\frac{\hbar\Omega}{2kT}\right). \tag{77}$$

10. Conclusions

We have shown that the perturbation series can be summed up to all orders if the commutator (19) is a c-number function of time. The response of the environment on the oscillator is then completely described by this function, and this function determines a certain class of environments of exactly solvable models. For practical applications the most important environment belonging to this class is the one described by oscillators coupled linearly with the original system oscillator (Haken 1970, Dekker 1981).

A discussion on the poles of the Laplace transformed $LA(s)$, given by equation (38), led to conditions for dissipative effects of the environment. We found that, for the homogeneous solution of $Q(t)$ approaching zero for large times, $LA(s)$ should have poles at the left half of the complex s plane only. Under this condition an approximative equation (63) of a damped oscillator is obtained. It is straightforward to calculate variances and correlation functions for the oscillator in the framework of the theory presented here. The results are naturally the same as for the known exactly solvable models, since the class of environments considered in this paper comprehend all known cases given in literature (see e.g. Dekker 1981).

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